Morse Theory Intro

2/3

Benjamin Thompson

- We'll assume manifolds are differentiable (all transition maps are smooth)

Def (Critical point)

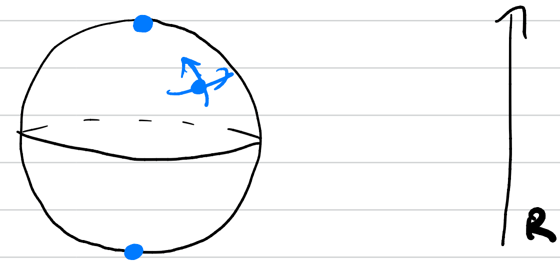
Let $f: M \to \mathbb{R}$ be a map of manifolds. Then the set of critical points of f is

$$\boxed{\mathrm{Crit} f = \{p \in M : df_p = 0\}}$$

Ex $S^2 \xrightarrow{\varphi} \mathbb{R}$

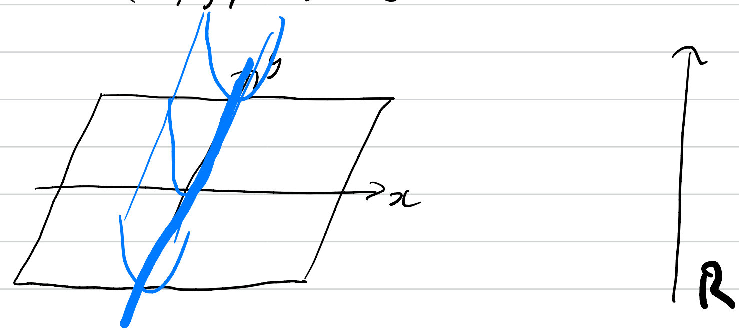
$(x,y,z) \overset{\varphi}{\mapsto} z$

$\mathrm{Crit}\,\varphi = \{(0,0,\pm 1)\}$



Ex $\mathbb{R}^2 \xrightarrow{\psi} \mathbb{R}$ $\quad \mathrm{Crit}\,\psi = \{y\text{-axis}\}$

$(x,y) \overset{\psi}{\mapsto} x^2$

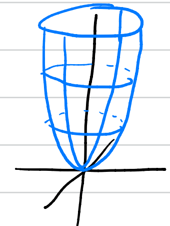


Def: A function $f : M \to \mathbb{R}$ is a Morse function if for every critical point p, there exists a chart around p so that f is locally

$$\boxed{f(x) = f(p) - x_1^2 - \cdots - x_k^2 + x_{k+1}^2 + \cdots + x_n^2.}$$

k is the index of p.

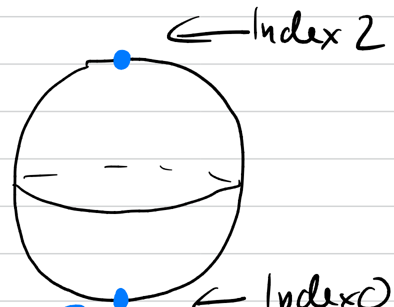
Paraboloid



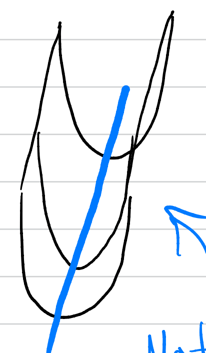
$z = x^2 + y^2$

Saddle

$z = x^2 - y^2$



after a coordinate change, $\approx x^2 + y^2$



Not a Morse function

Def (Hessian): For $V, W \in T_pM$, and $p \in \text{Crit}\, M$, define a bilinear symmetric form

$$H_p(f)(V,W) := V \cdot (\tilde{W} \cdot f)_p$$

where $\tilde{W}$ is an extension of W into a vector field at p.

If $H_p(f)$ is non-degenerate, p is called a non-degenerate (critical) point.

Morse Lemma: Let $p \in M$ be critical and non-degenerate with respect to $f: M \to \mathbf{R}$. Then there exists a smooth chart $\phi: U \to \mathbf{R}^m$, $\phi(p) = 0$, and

$$(f \circ \phi^{-1})(x_1, \dots, x_m) = f(p) - x_1^2 - \dots - x_k^2 + x_{k+1}^2 + \dots + x_m^2.$$

(So if all critical points of f are non-degenerate, then f is a Morse function.)

Palais' proof of the Morse Lemma:

- Replace f with $f - f(p)$, and choose a coordinate chart so that:

$$f(0) = 0, \quad df(0) = 0 \text{ and}$$

$$H_0(f) = \mathrm{Diag}(\underbrace{-1, -1, \dots, -1}_{k}, 1, 1, \dots, 1).$$

$H_0(f)$ induces a function $A : \mathbf{R}^m \to \mathbf{R}$ by

$$A(x) = \langle H_0(f)x, x\rangle.$$

We want to prove there is a diffeo $\varphi : U \to U'$ s.t.

$$f \circ \varphi = A.$$

(We want to construct the local coordinates.)

Moser's idea: define

$$f_t = A + t(f - A),$$

and search for φ_t satisfying:

$$f_t \circ \varphi_t = A.$$

(So $\varphi_t = \mathrm{id}$ solves this at $t=0$.)

Assume we knew what φ_t was. Then we could define curves $t \mapsto \varphi_t(x)$, and let ξ_t be the tangent to $t \mapsto \varphi_t(x)$.

Then we can write the derivative of φ_t as:

$$\frac{d\varphi_t}{dt}(x) = \xi_t(\varphi_t(x)) .$$

Differentiating $f_t \circ \varphi_t = A$ gives

$$(\dot{f}_t \circ \varphi_t + (\xi_t \cdot f_t) \circ \varphi_t)(x) = 0.$$

or

$$(\dot{f}_t + \xi_t \cdot f_t)(y) = 0.$$

$f_t = A + t(f - A)$, so

$\dot{f}_t = f - A$, hence

$$df_t(\xi_t) = A - f .$$

A scholium turns this into

$$\langle B_x^t \xi_t, x \rangle = \langle G_x x, x \rangle .$$

Finally integrate* $\xi_t = (B_x^t)^{-1} G_x x$ to get family of φ_t.

$\square$

Palais' Proof reference:

Augustin Banyaga &

David Hurtubise.

Scholium basically Hadamard's

lemma.