

CONSEQUENCES OF MORSE HOMOLOGY

Let $(M; M_0, M_1)$ be oriented & n -dim.

A) MORSE INEQUALITIES:

If $f: M \rightarrow [0, 1]$ is Morse, then
 $C_k \geq \text{rank } H_k(M, M_0)$

B) LEFSCHETZ DUALITY:

$$H_k(M, M_0) \cong H^{n-k}(M, M_1)$$

C) SMALE'S THEOREM: ($n \geq 6$)

If M, M_0, M_1 have $\pi_1 = 0$
and $H_k(M, M_0)$ is free $\forall k$,
we can get equality in (A).

\cong diffeo

\approx homeo

\sim hom equiv

IMPORTANT LEMMAS

D) SPHERE LEMMA:

For any $f \in \text{Homeo}(S^{n-1})$, $D^n \cup_f D^n \cong S^n$.

E) DISK LEMMA:

Any 2 orientation-preserving embeddings $D^n \hookrightarrow M$ are ambiently isotopic.

F) KERVAIRE, MILNOR, WALL: $(n=4, 5, 6)$

If $\Sigma \sim S^n$, then Σ bounds a contractible manifold.

G) CONNECTED SUM:

• If $\Sigma \sim S^n$, then $\Sigma \# (-\Sigma)$ bounds a contractible manifold.

• If $M \# N \cong S^n$, then $M \cong N \cong S^n$.

IMMEDIATE RESULTS

H) H-COBORDISM THM: ($n \geq 6$)

If M, M_0, M_1 have $\pi_1 = 0$ and the inclusion $M_0 \hookrightarrow M$ is a hom equiv, then $M \cong M_0 \times [0, 1]$.

I) CLASSIFICATION OF D^n : ($n \geq 6$)

If M is contractible and $\pi_1(\partial M) = 0$, then $M \cong D^n$.

J) GENERALIZED POINCARÉ CONT: ($n \geq 5$)

If $\Sigma \sim S^n$, then $\Sigma \cong S^n$.

For $n=5$ or 6 , we get $\Sigma \cong S^n$.

K) CHARACTERIZATION OF D^5 :

If M is contractible and

$\partial M \cong S^4$, then $M \cong D^5$.

GROUPS OF HOMOTOPY SPHERES

L) Closed, connected n -manifolds form a monoid under $\#$.

Let A^n be the group of invertibles.

For $n \geq 5$, $A^n = \{\text{smooth structures on } S^n\}$.

M) For any $f \in \text{Diff}(S^{n-1})$, let $p(f) = D^n U_f D^n$.
If $f \sim g$, then $p(f) \cong p(g)$.

$$\begin{array}{ccccccc} & & \text{Diff}(D^n) & & & & \\ & & \downarrow & & & & \\ 0 & \rightarrow & \text{Diff}_0(S^{n-1}) & \rightarrow & \text{Diff}(S^{n-1}) & \rightarrow & \text{MCG}(S^{n-1}) \rightarrow 0 \\ & & & & \downarrow p & \swarrow \bar{p} & \\ & & & & A^n & & \end{array}$$

The vertical maps are exact.

$n=5 \Rightarrow p$ is onto

$n=4 \Rightarrow p$ is 0 (Cerf)

Milnor and Kervaire relate these groups to stable homotopy groups of spheres.

VERSIONS OF SMOOTH SCHOENFLIES:

N) If $\Sigma \sim S^{n-1}$, then any embedding $(n \neq 4, 5)$
 $\Sigma \hookrightarrow S^n$ cuts out two standard disks.

O) Any embedding $S^{n-1} \hookrightarrow S^n$ cuts out
two standard disks. $(n \neq 4)$

P) If $\Sigma \sim S^{n-1}$ embeds into S^n , then $\Sigma \cong S^{n-1}$.
 $(n \neq 5)$

LOW DIMENSIONS:

Q) If $\Sigma \sim S^n$, then $\Sigma \cong S^n$. $(n = 1, 2, 3, 5, 6)$

R) Any $\Sigma \sim S^n$ admits a Morse function
with only two critical points. $(n \neq 4)$

Freedman: If $\Sigma \sim S^4$, then $\Sigma \cong S^4$.

Perelman: Everything works for $n=3!$



But open questions remain...

EQUIVALENCE EXERCISES

1. Use $Q(3)$ to prove everything else for $n=3$.

2. Prove the following implications between open questions for $n=4,5$:

